

Thermal ground state in Yang-Mills thermodynamics

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Abstract. We derive an a useful priori estimate for the thermal ground state of deconfining phase of SU(2) Yang-Mills thermodynamics in four-dimensional, flat spacetime and discuss its implications. Upon a selfconsistent spatial coarse-graining over noninteracting, trivial-holonomy (BPS saturated) (anti)calorons of unit topological charge modulus an inert, adjoint scalar field $|\phi|$ and an effective pure-gauge configuration a_μ^{gs} emerge. The modulus $|\phi| > 0$ defines the maximal resolution in the coarse-grained theory and induces dynamical gauge-symmetry breaking. Thanks to perturbative renormalizability and the fact that $|\phi|$ can not absorb or emit energy-momentum the effective action is local and simple. The temperature dependence of the effective coupling is a consequence of thermodynamical consistency and describes the Coulomb screening of a static test charge due to short-lived monopole-antimonopole pairs. The latter occur unresolvably as small-holonomy excitations of (anti)calorons by the absorption of propagating fundamental gauge fields.

Keywords: Trivial-holonomy calorons, Bogomoln'yi-Prasad-Sommerfield saturation, winding gauge, unitary gauge, perturbative renormalizability, physical gauge, effective gauge coupling, Coulomb screening, phase transition, nonperturbative asymptotic freedom, radiative corrections

PACS: 11.10.Wx, 11.15.Tk, 11.55.Fv, 02.60.Cb

INTRODUCTION

A purely perturbative approach to high-temperature Yang-Mills thermodynamics is inappropriate because the screening effects in the static magnetic sector are too weak to enforce the convergence of the expansion after resummation of low-order polarization effects [1]. On the other hand, Euclidean 4D SU(2) Yang-Mills theory disposes of topologically stabilized, (anti)selfdual field configurations [2, 3, 4, 5, 6, 7] which, due to a weight $\propto e^{-\frac{8\pi^2|k|}{g^2}}$ (g the coupling constant, k the topological charge) in the partition function, are ignored by small-coupling expansions. All finite-action (anti)selfdual field configurations ($F_{4i} = E_i = \pm \frac{1}{2}\epsilon_{ijk}F_{jk} = \pm B_i$, ($i, j, k = 1, 2, 3$), $F_{\mu\nu} = F_{\mu\nu}^a t^a \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ the field strength tensor, $A_\mu = A_\mu^a t^a$ the gauge field, and t^a , ($a = 1, 2, 3$), generators of SU(2) normalized to $\text{tr } t^a t^b = \frac{1}{2}\delta^{ab}$) share the property that their energy-momentum tensor $\Theta_{\mu\nu}$ vanishes identically:

$$\begin{aligned} \Theta_{\mu\nu} = & -2\text{tr}\left\{\delta_{\mu\nu}\left(\mp\vec{E}\cdot\vec{B}\pm\frac{1}{4}(2\vec{E}\cdot\vec{B}+2\vec{B}\cdot\vec{E})\right)\mp(\delta_{\mu 4}\delta_{\nu i}+\delta_{\mu i}\delta_{\nu 4})(\vec{E}\times\vec{E})_i\right. \\ & \left. \pm\delta_{\mu i}\delta_{\nu(j\neq i)}(E_i B_j - E_j B_i) \pm \delta_{\mu(j\neq i)}\delta_{\nu i}(E_j B_i - E_i B_j)\right\} \equiv 0, . \end{aligned} \quad (1)$$

To exploit this observation is essential in the construction of the thermal ground-state estimate in the deconfining phase of the theory: An effective field that is obtained by coarse-graining over noninteracting gauge-field configurations of vanishing pressure and energy density can not propagate!

PRINCIPLES OF INFINITE-VOLUME THERMODYNAMICS

Important constraints in constructing a useful estimate for the thermal ground state are imposed by thermodynamics itself. Here we state two basic principles of the infinite-volume case. These principles guide the coarse-graining process that involves fundamental, action minimizing gauge-field configurations of vanishing energy-momentum. They are:

I) In the absence of external sources, a thermodynamical gauge system in the infinite-volume limit guarantees, up to admissible gauge transformations, the spatial isotropy and homogeneity of an *effective* local field if this field is *not* associated with the propagation of energy-momentum by *fundamental* gauge fields: The partial ensemble average in combination with a spatial average, leading to the emergence of such a field, yields a nonzero result only if this field

is a rotational scalar ϕ , and ϕ does not depend on space up to admissible gauge transformations.

II) Up to admissible gauge transformations the field ϕ must not depend on time.

Principle I) is selfevident, principle II) is a consequence of fact that ϕ does not possess energy-momentum at any time.

CALORONS

The only (anti)selfdual or Bogomoln'yi-Prasad-Sommerfield (BPS) saturated fundamental gauge-field configurations of SU(2) Yang-Mills theory at high temperature T , which enter into the coarse-graining process leading to the emergence of the effective scalar field ϕ , turn out to be (anti)calorons [5, 4, 6, 7] of unit topological charge modulus [8, 9]. Calorons are instantons of period $\beta \equiv \frac{1}{T}$ in Euclidean time τ . If at spatial infinity the exponential of the integral of iA_4 over one period (the Polyakov loop) coincides with an element of the center group \mathbf{Z}_2 then the caloron is said to be of trivial holonomy. The trivial-holonomy charge-one caloron, which is stable under radiative corrections [10] and thus enters into the a priori estimate of the thermal ground state in terms of the field ϕ , was constructed in [4] based on the work [11, 12] (particular multiinstantons). In singular gauge, where topological charges are located at instanton centers, the trivial-holonomy expressions for a charge-modulus-one caloron (C) or anticaloron (A) read:

$$\bar{A}_\mu^{C,a}(x) = -\bar{\eta}_{\mu\nu}^a \partial_\nu \log \Pi, \quad \bar{A}_\mu^{A,a}(x) = -\eta_{\mu\nu}^a \partial_\nu \log \Pi, \quad (2)$$

where

$$\Pi(\tau, \vec{x}; \rho, \beta) = 1 + \frac{\pi \rho^2}{\beta r} \frac{\sinh\left(\frac{2\pi r}{\beta}\right)}{\cosh\left(\frac{2\pi r}{\beta}\right) - \cos\left(\frac{2\pi \tau}{\beta}\right)}, \quad (3)$$

$r \equiv |\vec{x}|$, $\eta_{\mu\nu}^a$ ($\bar{\eta}_{\mu\nu}^a$) denotes the selfdual (antiselfdual) 't Hooft symbol, $\rho > 0$ is the size parameter, and the configuration possesses no overall magnetic charge. Variability of ρ expresses the invariance of the classical Yang-Mills action $\text{tr} \frac{1}{2} \int d\tau d^3x F_{\mu\nu} F_{\mu\nu}$ under spatial scale transformations. For reasons mentioned below we do not make other moduli of these minimal-action configurations explicit. The much harder construction of a nontrivial-holonomy charge-one caloron (no overall magnetic charge but magnetic-monopole constituents) is based on the Nahm transformation and was carried out explicitly in [6, 7]. While a configuration with static holonomy has vanishing quantum weight in the infinite-volume limit [13] and thus does not contribute to the partition function the concept of a temporary holonomy, associated with short-lived magnetic dipoles (small holonomy) or stable, screened magnetic (anti)monopoles (large holonomy), nevertheless applies.

INERT, ADJOINT SCALAR FIELD

The (dimensionless) phase $\{\hat{\phi}\}$ of the field $\phi = \{\hat{\phi}\}|\phi|$ ($|\phi| \equiv \sqrt{\frac{1}{2} \text{tr} \phi^2}$), which due to Lie-algebra valued gauge fields in the fundamental Yang-Mills action must be an adjointly transforming under gauge rotations, is contained in the set defined as

$$\{\hat{\phi}^a\} \equiv \sum_{C,A} \text{tr} \int d^3x \int d\rho t^a F_{\mu\nu}(\tau, \vec{0}) \left\{ (\tau, \vec{0}), (\tau, \vec{x}) \right\} F_{\mu\nu}(\tau, \vec{x}) \left\{ (\tau, \vec{x}), (\tau, \vec{0}) \right\}, \quad (4)$$

where

$$\left\{ (\tau, \vec{0}), (\tau, \vec{x}) \right\} \equiv \mathcal{P} \exp \left[i \int_{(\tau, \vec{0})}^{(\tau, \vec{x})} dz_\mu A_\mu(z) \right], \quad \left\{ (\tau, \vec{x}), (\tau, \vec{0}) \right\} \equiv \left\{ (\tau, \vec{0}), (\tau, \vec{x}) \right\}^\dagger. \quad (5)$$

Since a priori no spatial scale is available on the level of BPS saturation and because of spatial isotropy and homogeneity the Wilson lines in Eq. (5) need to be computed along the straight spatial line connecting the points $(\tau, \vec{0})$ and (τ, \vec{x}) , and \mathcal{P} demands path-ordering symbol. In (4) the sum is over the $|k| = 1$ caloron (C) and anticaloron (A) of trivial holonomy.

It can be shown [8, 9, 14] that the definition of the set in (4) is unique: no higher nonlocal n -point products with factors $F_{\mu\nu}$ and no (anti)calorons of higher topological charge modulus may contribute. This set turns out to be the kernel of a uniquely defined linear differential operator \mathcal{D} . Notice that the kernel and thus the associated operator would, due to the (anti)selfduality of the gauge field A_μ , be trivial if it were defined in a *local* way [8].

The evaluation of the right-hand side of (4) is quite intricate and subtle. In performing it one notices that only the magnetic-magnetic correlation contributes in the sense that a divergence of the r integration is multiplied by a zero of the angular azimuthal integration to yield an undetermined real scale factor for each polarization state in the azimuthal plane. Also, there is a freedom of choice of phase for each polarization. Together this gives a four-dimensional real parameter space for oscillatory motion in the plane, two parameters for each oscillation. Thus $\mathcal{D} = \partial_\tau^2 + \left(\frac{2\pi}{\beta}\right)^2$. The apparent breaking of rotational symmetry introduced by the angular regularization is unmasked as a global choice of gauge, there is a fast saturation of the set to the kernel of \mathcal{D} in the infrared, and no ultraviolet divergence arises. The fact that differential operator \mathcal{D} is linear and that $|\phi| \equiv \sqrt{\frac{1}{2}\text{tr}\phi^2}$ does not depend on space and time implies that \mathcal{D} annihilates the entire field $\phi = |\phi|\hat{\phi}$. By virtue of the Euler-Lagrange equations this in turn implies that the field ϕ possesses a canonic kinetic term $\text{tr}(\partial_\tau\phi)^2$ in its effective, Euclidean Lagrangian density \mathcal{L}_ϕ . Moreover, the Euler-Lagrange equations for ϕ and the (anti)selfduality (BPS saturation) of (anti)calorons imply that the explicit β dependence in \mathcal{D} be replaced by the ϕ -derivative of a potential $V(\phi^2)$. Therefore

$$\mathcal{L}_\phi = \text{tr}((\partial_\tau\phi)^2 + V(\phi^2)). \quad (6)$$

We now determine $V(\phi^2)$. The Euler-Lagrange equations derivable from Eq. (6) read [15]

$$\partial_\tau^2\phi^a = \frac{\partial V(|\phi|^2)}{\partial|\phi|^2}\phi^a \text{ (in components)} \Leftrightarrow \partial_\tau^2\phi = \frac{\partial V(\phi^2)}{\partial\phi^2}\phi \text{ (in matrix form)}. \quad (7)$$

Since ϕ 's motion is within a 2D plane in $\text{su}(2)$, since $|\phi|$ is independent of space and time, and since ϕ 's phase $\hat{\phi}$, viewed as a function of τ , is of period β , one may, without restriction of generality write the solution to Eq. (7) as

$$\phi = 2|\phi|t_1 \exp(\pm \frac{4\pi i}{\beta}t_3\tau). \quad (8)$$

BPS saturation and in particular the vanishing of the Euclidean energy density together with Eq. (8) imply that

$$|\phi|^2 \left(\frac{2\pi}{\beta}\right)^2 - V(|\phi|^2) = 0. \quad (9)$$

Comparing Eq. (7) with $\partial_\tau^2\phi + \left(\frac{2\pi}{\beta}\right)^2\phi = 0$, we have

$$\left(\frac{2\pi}{\beta}\right)^2 = -\frac{\partial V(|\phi|^2)}{\partial|\phi|^2}. \quad (10)$$

Together, Eqs. (9) and (10) yield

$$\frac{\partial V(|\phi|^2)}{\partial|\phi|^2} = -\frac{V(|\phi|^2)}{|\phi|^2} \Rightarrow V(|\phi|^2) = \frac{\Lambda^6}{|\phi|^2} \Rightarrow |\phi| = \sqrt{\frac{\Lambda^3\beta}{2\pi}}, \quad (11)$$

where the mass scale Λ is a constant of integration which can be used to define a dimensionless temperature: $\lambda \equiv \frac{2\pi T}{\Lambda}$. In contrast to perturbation theory, where the Yang-Mills scale is associated with the Landau pole for the running gauge coupling g dimensional transmutation in the effective theory is free of such a contradiction.

To arrive at the entire effective action valid at maximal resolution $|\phi|$ one exploits that perturbative renormalizability [16, 17, 18, 19] implies that coarse-graining out topologically trivial, propagating quantum fluctuations down to resolution $|\phi|$ does not alter the form of the action density of the associated effective modes. Moreover, gauge invariance demands that $\partial_\tau\phi \rightarrow D_\mu\phi = \partial_\mu\phi - ie[a_\mu, \phi]$ where a_μ denotes the effective, propagating gauge field, and e is the effective coupling constant. Also, the inertness of the field ϕ due to BPS saturation does not allow for momentum transfer mediated by local, higher-dimensional operators, and therefore the effective action density reads

$$\mathcal{L}_{\text{eff}}[a_\mu] = \text{tr} \left(\frac{1}{2} G_{\mu\nu} G_{\mu\nu} + (D_\mu\phi)^2 + \frac{\Lambda^6}{\phi^2} \right), \quad (12)$$

where $G_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - ie[a_\mu, a_\nu] \equiv G_{\mu\nu}^a t_a$ denotes the effective field strength, and the full ground-state estimate at tree-level is given by ϕ and by the pure-gauge configuration $a_\mu^{\text{gs}} = \mp \delta_{\mu 4} \frac{2\pi}{eB} t_3$. The latter solves the Euler-Lagrange equations for a_μ in the effective theory. Notice that a_μ^{gs} generates transmutates vanishing pressure and energy density of the field ϕ into negative and positive values of the ground-state estimate. Microscopically, this relates to an interaction-induced temporary, small holonomy for each (anti)caloron implying the creation, subsequent collapse, re-creation, ... of monopole-antimonopole pairs. A gauge rotation to unitary gauge $\phi = \delta^{3a} |\phi|$ reveals the breaking of the electric \mathbf{Z}_2 symmetry associated with a finite expectation of the Polyakov loop [9, 14]. This confirms the deconfining nature of the discussed phase.

THERMAL QUASIPARTICLES ON TREE LEVEL

The effective action density \mathcal{L}_{eff} in (12) has an in-built mechanism for gauge symmetry breaking $\text{SU}(2) \rightarrow \text{U}(1)$ due to $|\phi| > 0$. As a consequence, two out of three $\text{su}(2)$ directions of propagating gauge modes acquire mass on tree-level. The requirement that this mass is thermodynamically consistent, namely, that, at the one-loop level, the energy density of the effective theory can be obtained by a Legendre transformation from the effective pressure yields the following evolution equation for e

$$1 = -\frac{24\lambda^3}{(2\pi)^6} \left(\lambda \frac{da}{d\lambda} + a \right) a D(2a), \quad (13)$$

where $a \equiv \frac{m}{2T}$, $m = 2e\sqrt{\Lambda^3/(2\pi T)}$, and $D(y) \equiv -\frac{4\pi^2}{T^4} \partial_{y^2} P = \int_0^\infty dx \frac{x^2}{\sqrt{x^2+y^2}} \frac{1}{e^{\sqrt{x^2+y^2}} - 1}$. There are two fixed points of this evolution: One at $\lambda_c \equiv 13.87$, where e exhibits a logarithmic pole, $e(\lambda) \propto -\log(\lambda - \lambda_c)$, and one for $\lambda \rightarrow \infty$ where $a \rightarrow 0$. A plateau value $e = \sqrt{8}\pi$ is reached rapidly as λ increases away from λ_c . This expresses the fact that the strength of the $\text{U}(1)$ Coulomb field of an isolated magnetic test charge is renormalized independently of temperature: a nonperturbative manifestation of asymptotic freedom. Effective radiative corrections introduce a mass scale into Coulomb-field screening. This is due to the dissociation of (anti)calorons into stable monopole-antimonopole pairs through the excitation of large holonomies: The magnetic field of dynamically created, stable (anti)monopoles is short range [22].

OUTLOOK ON RADIATIVE CORRECTIONS

The computation of effective radiative corrections in real-time signature is explained in detail in the talks by Markus Schwarz [20] and Dariush Kaviani [21]. Here we just spell out the main features of the effective loop expansion. In [23] it is conjectured that the expansion into 1-particle irreducible diagrams of the polarization tensors of each gauge mode (reducible diagrams are resummed) terminates at finite orders. This carries over to the loop expansion of any thermodynamical quantity, say the pressure. The argument uses the fact that nonempty intersections of the algebraic varieties defined by the constraints on the loop variables imposed by the maximal resolution $|\phi|$ become extremely unlikely at a sufficiently large number of (independent) loops. It is these intersections, however, that support the loop integrations. To make this plausible, the number of constraints on a priori noncompact loop variables as a function of loop number can be estimated by appealing to the Euler- L' Huilliers characteristics of the polyhedron associated with a given loop diagram. One observes that the number of constraints starts to exceed the number of loop variables very early in the loop expansion. Numerically, we investigated the two- and three-loop situation in the pressure expansion, and we see very fast convergence.

SUMMARY

This talk is on the construction of a thermal ground-state estimate for deconfining $\text{SU}(2)$ Yang-Mills thermodynamics. In a first step, this construction invokes a spatial coarse-graining over noninteracting (anti)calorons of unit charge modulus and trivial holonomy. The associated emergence of an inert, adjoint scalar field ϕ and perturbative renormalizability imply a simple, local effective action valid for a maximal resolution $|\phi|$. We derive a tree-level ground-state estimate for this effective theory and discuss its implications for the spectrum and the evolution of the effective coupling. Finally, we give a brief outlook on effective radiative corrections.

ACKNOWLEDGMENTS

I would like to express my gratitude to the following Diploma students at the universities of Heidelberg and Karlsruhe and the Karlsruhe Institute of Technology who, over a period of seven years, helped to develop the ideas and results discussed in this symposium: Ulrich Herbst, Jochen Rohrer, Markus Schwarz, Dariush Kaviani, Michal Szopa, Jochen Keller, Sebastian Scheffler, Julian Moosmann, Josef Ludescher, and Carlos Falquez. To Francesco Giacosa, who, in an exemplary way and under difficult conditions, committed himself to common research and to sober, constructive criticism, go my very special thanks. I feel indebted to Theodore Simos for his suggestion to organize this meeting and for his outstanding efforts in running the International Conferences on Numerical Analysis and Applied Mathematics.

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